2.5: A Closer Look at the Euler Method

Theorem 1. (Error in the Euler Method)
Suppose that the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

has a unique solution $y(x)$ on the closed interval $[a, b]$ with $a = x_0$ and assume that y has a continuous second derivative on $[a, b]$. Then there exists a constant $C \in \mathbb{R}$ such that if $y_1, \ldots, y_k$ are the approximations (with the Euler Method) to the actual values $y(x_1), \ldots, y(x_k)$, then

$$|y_n - y(x_n)| \leq Ch,$$

where $h > 0$ is the step size and this holds for all $n = 1, 2, \ldots, k$.

The Improved Euler Method. Given the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0,$$

the **improved Euler method with step size** h consists in applying the iterative formulas

$$\begin{aligned} k_{n,1} &= f(x_n, y_n) \\ u_{n+1} &= y_n + h \cdot k_1 \\ k_{n,2} &= f(x_{n+1}, u_{n+1}) \\ y_{n+1} &= y_n + h \cdot \tfrac{1}{2}(k_1 + k_2) \end{aligned}$$

to compute successive approximation $y_1, y_2, y_3, \ldots$ to the true values $y(x_1), y(x_2), y(x_3), \ldots$.

Exercise 1. Use the improved Euler method to approximate the solution to

$$\frac{dy}{dx} = x + y, \quad y(0) = 1$$

with step size $h = 0.1$ on the interval [0,0.5]. Given that the exact solution is $y(x) = 2e^x - x - 1$ find the error in this approximation.

Exercise 1. (Continued)

Rutte-Kunge Method. See section 2.6 for another approximation known as the Rutte-Kunge Method.

Homework. 1-5 (odd)